HOME WORK V, HIGH-DIMENSIONAL GEOMETRY AND PROBABILITY, SPRING 2018

Due April 3 (extended deadline, long break, hence bigger home work:)). All the questions marked with an asterisk(s) are optional. The questions marked with a double asterisk are not only optional, but also have no due date.

Question 1**.** Show that in general (non-symmetric) case, the maximal volume ratio of K is achieved when K is a simplex.

Question 2^***.** Let $Q=[-\frac{1}{2},\frac{1}{2}]^n\subset\mathbb{R}^n$ and let H be an $(n-k)$-dimensional subspace. Prove that $|Q\cap H|_{n-k}\geq 1$.

Question 3**.** Let $Q=[-\frac{1}{2},\frac{1}{2}]^n\subset\mathbb{R}^n$ and let H be an $(n-k)$-dimensional subspace. Show an alternative upper bound for the slices of the cube:

$$|Q\cap H|_{n-k}\leq\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}};$$

for what values of k is it an improvement upon $\sqrt{2}^k$? What does it approach as $k\approx n$?
Hint 1: Consider the dual situation: project onto H, and let $w_i=e_1|_H$; show that

$$|Q\cap H|_{n-k}=\int_H\prod_i 1_{[-\frac{1}{2},\frac{1}{2}]}(\langle x,w_i\rangle)dx.$$

Hint 2: Suppose $c_i\geq 0$ for $i=1,...,n$ and $\sum c_i=A$. Show that

$$\prod_1^n c_i^{c_i}\geq\prod_1^n\left(\frac{A}{n}\right)^{\frac{A}{n}}.$$

Question 4^***.** Let θ be a random vector uniformly distributed on $\mathbb{S}^{n-1}$. Find $\mathbb{E}|Q\cap\theta^\perp|_{n-1}$, up to $1+o(1)$ multiplicative factor.

Question 5**.** Suppose a convex body K contains a ball of radius $R > 0$. Show that

$$\gamma^+(\partial K) \leq \frac{\sqrt{n}}{R}.$$

Hint: How to bound the support function from below in this case?

Question 6**.** Suppose C is a convex cone with a vertex at the origin.

a) Show that $\gamma^+(\partial C) \leq 10$.

$b)^{**}$ Find the exact constant $\sup_C \gamma^+(\partial C)$.

Hint: How to bound the support function from above in this case?

Question 7^***.** Let $P = \prod_{i=1}^n [-\frac{x_i}{2}, \frac{x_i}{2}]$ be a regular parallelepiped.

a) Find $\max_{u \in \mathbb{S}^{n-1}} |P \cap u^\perp|_{n-1}$.

$b)^{**}$ Find $\max_H |P \cap H|_{n-2}$, where the maximum runs over subspaces H of co-dimension 2.

Question 8^{**}**.** For convex regions $K \subset \mathbb{R}^2$, find *exactly* $\gamma^+(\partial K)$.

Question 9^{**} **(credit goes to Josiah Park).** How big, and how small can $\frac{vr(K)}{d_{BM}(K)}$ be? You may assume symmetry.